# DYNAMIC COUPLED PROBLEM OF THERMOELASTICITY FOR A HALF-SPACE TAKING ACCOUNT OF THE FINITENESS OF THE HEAT PROPAGATION VELOCITY

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The coupled problem of thermoelasticity for a half-space in the case of an infinite heat propagation velocity has been considered in [1 to 4]. It was found there: the solution for small times by an expansion in the small coupling parameter $\delta$; the asymptotic behavior of the solution between the surface of the half-space and the acoustic wave front for large times; and also the value of the jumps in the various quantities and their derivatives at the acoustic wave front. The same set of questions is examined herein, but the asymptotic behavior of the solution for large times is studied in the whole domain of existence of the solution. For small and moderate times ($t\delta \sim 1$) the maximum stress is achieved at the accoustic wave front. This maximum decays exponentially, and depends on the velocity of heat propagation. A second maximum, whose propagation velocity and magnitude are independent of the velocity of heat wave propagation, is manifest at considerable times ($t\delta \gg 1$). This maximum decays as $\sim 1/\sqrt{t}$, hence, it yields a fundamental contribution to the state of stress at very large times.

Taking account of the finiteness of the heat propagation velocity, the heat conduction equation is discussed and derived in [5 to 8].

A term taking account of the inertia of the thermal flux

$$\tau \frac{\partial q}{\partial t} + q = -\lambda \frac{\partial T}{\partial x} \tag{1}$$

appears in the Fourier law.

The energy conservation Eq. [9], the equation of motion in the acoustic approximation, and Hooke's law are

$$\rho c_v \frac{\partial T}{\partial t} + \rho \frac{c_p - c_v}{\alpha} \frac{\partial^2 u}{\partial t\, \partial x} = -\frac{\partial q}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \qquad \frac{\partial u}{\partial x} = \frac{1}{1-\mu}\left[\frac{1-2\mu}{2G}\sigma + (1+\mu)\alpha T\right] \tag{2}$$

Here $T$ is the temperature; $q$ the heat flux; $u$ the displacement in the $x$ direction perpendicular to the half-space surface ($x > 0$); $\sigma$ the normal stress in a plane parallel to the surface; $t$ the time; $\tau$ the relaxation time of the heat flux; $\lambda$ the heat conductivity; $\rho$ the density; $c_p$, $c_v$ the specific heats at constant pressure and volume, respectively; $\alpha$ the coefficient of thermal expansion; $\mu$ the Poisson coefficient; $G$ the shear modulus.

Let us reduce the system to nondimensional form by introducing the quantities

$$\frac{a}{c^2}, \quad \frac{a}{c}, \quad T_0, \quad 2\alpha G\frac{1+\mu}{1-2\mu}T_0, \quad \alpha T_0\frac{1+\mu}{1-\mu}\frac{a}{c}, \quad \lambda\frac{T_0 c}{a}$$

as the time, length, temperature, stress, displacement and heat flux scales, where $a$ is the coefficient of thermal diffusivity, amd $c$ the propagation velocity of the compression wave.

After having been reduced to nondimensional form, the system becomes (the nondimensional variables are now denoted by the same symbols)

$$\frac{1}{b^2}\frac{\partial q}{\partial t}+q=-\frac{\partial T}{\partial x}, \quad \frac{\partial T}{\partial t}+\delta\frac{\partial^2 u}{\partial x\,\partial t}=-\frac{\partial q}{\partial x}, \quad \frac{\partial^2 u}{\partial t^2}=\frac{\partial \sigma}{\partial x}, \quad \sigma=\frac{\partial u}{\partial x}-T$$

$$\left(\delta=\frac{(\varkappa-1)(1+\mu)}{1-\mu}, \quad \varkappa=\frac{c_p}{c_v}, \quad b=\frac{1}{c}\left(\frac{\lambda}{\rho c_v \tau}\right)^{1/2}\right) \tag{3}$$

where $b$ has the meaning of a nondimensional velocity of thermal perturbation for the uncoupled problem ($\delta=0$).

Stresses are absent in the half-space at the initial instant and the temperature is constant. $T$ is henceforth measured from precisely this initial level. For $t>0$ perturbations are given on the surface. There are no internal heat sources. Hence, if the velocity of all the perturbations is finite, then all the partial derivatives with respect to time are zero throughout at the instant $t=0$.

Applying the one-sided Laplace transform to the system (3), after simple manipulations we obtain

$$\frac{d^4\sigma^*}{dx^4}-\frac{d^2\sigma^*}{dx^2}\left[p^2\left(1+\frac{1+\delta}{b^2}\right)+p(1+\delta)\right]+\sigma^*\left(\frac{p^4}{b^2}+p^3\right)=0 \tag{4}$$

where the asterisk denotes the transform. We seek the solution in the form $\exp(kx)$. For $k$ we obtain (5)

$$k^2=\frac{1}{2}\left\{p^2\left(\frac{1+\delta}{b^2}+1\right)+p\left[1+\delta+\left(\left[p\left(\frac{1+\delta}{b^2}-1\right)+1+\delta\right]^2+\frac{4p^2\delta}{b^2}+4p\delta\right)^{1/2}\right]\right\}$$

The problem is considered in the $x>0$ half-space, hence, the general solution has the form

$$\sigma^*=c_1\exp(-k_+x)+c_2\exp(-k_-x)$$

Here $k_+$ and $k_-$ are the arithmetic roots corresponding to the solution which decays as $x\to+\infty$.

Let us consider the V. I. Danilovskaia problem

$$\sigma=0, \quad T=\eta(t) \quad \text{for } x=0, \quad \eta(t)=\begin{cases}1\ (t>0)\\ 0\ (t<0)\end{cases}$$

Using the third and fourth Eqs. of (3), we obtain for the transform

$$\sigma^*=0, \quad d^2\sigma^*/dp^2=0 \quad \text{for } x=0$$

After having satisfied the boundary conditions, the solution for the transform becomes

$$\sigma^*=p^2\frac{\exp(-k_+x)-\exp(-k_-x)}{k_+^2-k_-^2}$$

We obtain the original by means of the inversion formula

$$\sigma=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sigma^* e^{pt}\frac{dp}{p}=F(k_+)-F(k_-)$$

$$\left(F(k_\pm)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}G(k_\pm)\,dp, \quad G(k_\pm)=\frac{\exp(pt-k_\pm x)}{k_+^2-k_-^2}\right) \tag{6}$$

The function $G(k_+)$ is analytic in the half-space $\operatorname{Re} p > a$.
The expansion of $k_\pm$ in the neighborhood of $p = \infty$ is

$$k_\pm = \frac{p}{v_\pm} + \beta_\pm + \frac{\Phi_\pm}{p} + O(p^{-2})$$

$$\left(v_\pm = \left\{\frac{1}{2}\left[\left(\frac{1+\delta}{b^2}+1\right) \pm \left(\left(\frac{1+\delta}{b^2}-1\right)^2 + \frac{4\delta}{b^2}\right)^{1/2}\right]\right\}^{-1/2} \right. \tag{7}$$

$$\beta_\pm = \frac{v_\pm}{4}\left\{1+\delta \pm \frac{(1+\delta)^2/b^2 - 1 + \delta}{\sqrt{[(1+\delta)/b^2 - 1]^2 + 4\delta/b^2}}\right\}$$

$$\Phi_\pm = {}^1/_{32}v_\pm(\pm 4g - h_\pm^2 v_\pm^2), \quad h_\pm = 1+\delta \pm k_1, \quad g = \frac{(1+\delta)^2 - k_1^2}{k_0}, \quad k_1 = \frac{\gamma(1+\delta)+2\delta}{k_0}$$

$$k_0 = \sqrt{\gamma^2 + 4\delta/b^2}, \qquad \gamma = \frac{1+\delta}{b^2} - 1$$

i.e. the solution consists of two waves being propagated with the velocities $v_+$ and $v_-$. For $x > v_+ t$ we have $F(k_+) = 0$. The dependence of $v_+$ and $v_-$ on $b$ for $\delta = 0.073$ is pictured by solid lines in Fig. 1. The asymptotes pictured by dashes yield the wave velocities of the uncoupled problem ($\delta = 0$). The waves are hence separated into acoustic ($v_s = 1$) and thermal ($v_t = b$). There is no such separation for the coupled problem: the waves are divided into fast ($v_-$) and slow ($v_+$). For small $\delta$ it may provisionally be assumed that for $b < 1$ the acoustic wave ($v_s$) will be the fast wave ($v_-$), and for $b > 1$, the slow wave ($v_+$).

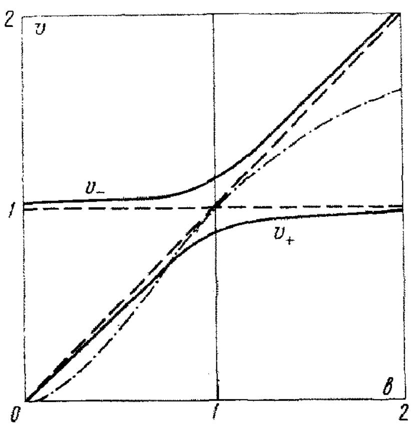


Fig. 1

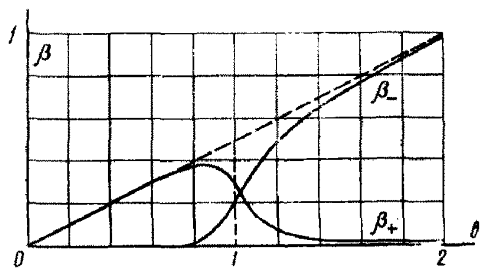


Fig. 2

It follows from (7) that $k_\pm$ is analytic in the neighborhood of $p = \infty$, and the point $p = \infty$ will be an isolated singular point for $G(k_\pm)$. Then by applying customary operational calculus methods, the contour of integration in (6) may be transformed into the neighborhood of the infinitely distant point and $\sigma$ may be evaluated by residue theory

$$\sigma = \operatorname*{res}_{p=\infty} G(k_+)\,\eta\left(t - \frac{x}{v_+}\right) + \operatorname*{res}_{p=\infty} G(k_-)\,\eta\left(t - \frac{x}{v_-}\right) \tag{8}$$

$$\operatorname*{res}_{p=\infty} G(k_\pm) = \left[\left(\frac{1+\delta}{b^2} - 1\right)^2 + \frac{4\delta}{b^2}\right]^{-1/2} \exp(-\beta_\pm x)\{1 +$$

$$+ \theta_\pm\left(\Phi_\pm x + \frac{1}{2}\frac{k_1}{k_0}\right) + \theta_\pm^2\left(\frac{\Phi_\pm^2 x^2}{2} + \frac{\Phi_\pm v_\pm^2 h_\pm x}{4} +\right.$$

$$\left. + \frac{\Phi_\pm x k_1}{k_0} + \frac{3}{8}\frac{k_1}{k_0^2} - \frac{1}{2}\frac{(1+\delta)^2}{k_0^2}\right) + \ldots\Big\}, \qquad \theta_\pm = t - \frac{x}{v_\pm}$$

Formula (8) yields a formal solution of the problem, however, the series in (8) converge poorly for large $\theta$ and $x$ and, in practice they may only be utilized in a few cases: for small $t$, in the neighborhood of the front of a fast wave (small $\theta$), and to compute the stress jumps on the wave fronts ($\theta_{\pm} = 0$) equal to

$$\exp(-\beta_{\pm} x)\left[\left(\frac{1+\delta}{b^2}-1\right)^2+\frac{4\delta}{b^2}\right]^{-1/2}$$

The dependence of the decay coefficients $\beta_{\pm}$ on $b$ is presented in Fig. 2 for $\delta = 0.017$ (solid curve) and $\delta = 0$ (dashed). In the neighborhood of the point $b = 1$ the phenomenon of resonance is observed, and the stress jumps at the fronts of both waves decay with the identical rate (for $\delta = 0$, $\beta_+^{(1)} = \beta_-^{(1)} = 1/4$). At the acoustic wave front, $\beta$ approaches the classical [1 and 2] value $\frac{1}{2}\delta$ sufficiently rapidly as $b \to \infty$.

To obtain more information about the solution it is necessary to express it in terms of the singularity of the transform in the finite part of the plane. The roots $k_+$, $k_-$ and $k_+^2 - k_-^2$

$$p_1 = 0, \quad p_2 = -b^2 \quad p_{3,4} = \frac{-\{(1/b^2) - 1 + \delta[1 + (2/b^2) + (\delta/b^4)]\} \pm i\sqrt{2\delta}}{[(1/b^2) - 1]^2 + (\delta/b^2)[2 + (2+\delta)/b^2]}$$

are possible branch points. If $p_1$, $p_2$ and $p_3$, $p_4$ are connected in pairs by slits, then $k_+$ and $k_-$ are analytic outside these slits, and are branches of a multivalued function. Three cases of the mutual disposition of the slits may be distinguished depending on $b$: the vertical slit (a) lies to the left of the horizontal slit $(p_1, p_2)$ $(b < 1)$, (b) intersects the horizontal $(b \sim 1)$, (c) lies to the right $(b > 1)$. The integral along the line $(a - i\infty, a + i\infty)$ in the inversion formula may be replaced by integrals along the slits traversed counter-clockwise [10]

$$\sigma = \frac{1}{2\pi i}\left\{\Phi(x,t) + \oint_{p_3}^{p_4}\left[G(k_+)\,\eta\left(t - \frac{x}{v_+}\right) - G(k_-)\,\eta\left(t - \frac{x}{v_-}\right)\right]dp\right\}$$

$$\Phi(x,t) = \oint_{p_1}^{p_2} G(k_+)\,\eta\left(t-\frac{x}{v_+}\right)dp, \qquad \left(\oint_{\mathrm{Re}\,p_3}^{p_2} G(k_+)\,dp\,\eta\left(t-\frac{x}{v_+}\right) - \right. \tag{9}$$

$$- \oint_{p_1}^{\mathrm{Re}\,p_3} G(k_-)\,dp\,\eta\left(t-\frac{x}{v_-}\right), \quad \left. - \oint_{p_1}^{p_2} G(k_-)\,dp\,\eta\left(t-\frac{x}{v_-}\right)\right)$$

for cases (a), (b) and (c), respectively.

The solution (9), written in the form of integrals, is useful in that the asymptotic expansion for large $x$ and $t$ may be obtained from it, and this is of greatest practical interest because of the smallness of the length and time scales utilized in defining the nondimensional quantities. Formula (9) may be simplified since the integrals of $G(k_+)$ and $G(k_-)$ along the vertical slit $(p_3, p_4)$ are equal. The fact is that the roots $k_+$ and $k_-$ change into each other upon passage through the vertical slit, and the root in the denominator of the integrand has different signs on the sides of the slit. Formula (9) becomes

$$\sigma = \frac{1}{2\pi i}\left\{\Phi(x,t) + \left[\eta\left(t-\frac{x}{v_+}\right) - \eta\left(t-\frac{x}{v_-}\right)\right]\oint_{p_3}^{p_4} G(k_-)\,dp\right\} \tag{10}$$

To obtain the asymptotic representation let us use the saddle point method. The saddle points will be the roots of the Eqs.

$$\partial f_{\pm} / \partial p = 0 \qquad (f_{\pm} = p - k_{\pm} v, \; v = x/t) \tag{11}$$

They are easily found for $\Phi$ at $\delta = 0$. In the case of an arbitrary parameter $\delta$ Eq.

(11) must be solved numerically. However, we can limit ourselves to small $\delta$ since $\delta$ does not customarily exceed 0.1. We hence obtain for the saddle point of $\Phi$

$$p_\pm = \frac{b^2}{2}\left(-1 \pm \frac{1}{\sqrt{1-v^2/b^2}}\right) + \tag{12}$$

$$+\frac{v^2\delta}{2\sqrt{1-v^2 b^2}}\left[\frac{(1-b^2)\sqrt{1-v^2/b^2}\pm(1+b^2)}{(1+b^2)\sqrt{1-v^2/b^2}\pm(1-b^2)} \pm \frac{1}{2(1-v^2/b^2)}\right] + O(\delta^2)$$

The largest value of $f$ is achieved at $p_+$, hence, the principal term of the asymptotic expansion of $\sigma$ becomes

$$\sigma = \frac{v}{\sqrt{\pi t}[1-b^2+(1+b^2)\sqrt{1-v^2/b^2}](1-v^2/b^2)^{1/4}}\left\{1-\delta\left[\frac{1}{4}\left(1-\frac{3}{A_0^2}\right)+\right.\right.$$

$$\left.\left.+\frac{v^2\gamma_0}{2A_2A_0^2}+\frac{3A_0+2}{2A_2}+\frac{4v^2-3A_1}{2A_2^2}+\frac{2\gamma_0 v^2 A_1}{A_2^3}\right]+O(\delta^2)\right\}\times \tag{13}$$

$$\times\exp\left[-\frac{b^2 t(1-\sqrt{1-v^2/b^2})}{2}-\frac{\delta t v^2}{4}\left(\frac{1}{A_0}+\frac{2}{A_2}\right)+O(\delta^2)\right]$$

$$(A_0=\sqrt{1-v^2/b^2},\quad A_1 = 1+b^2+(1-b^2)\sqrt{1-v^2/b^2}$$

$$A_2 = 1-b^2+(1+b^2)\sqrt{1-v^2/b^2},\quad \gamma_0=(1/b^2)-1)$$

$$\left[\left|\frac{b^2+b^4}{b^2-1}\left(v-\frac{2b^2}{1+b^2}\right)\right| \gg \max\left(\delta,\frac{1}{\sqrt{t}}\right)\right],\quad \left[0<v<\min\left(\frac{2b^2}{1+b^2},\ v_+\right)\right]$$

$$\left[\frac{2b^2}{1+b^2}<v<v_-, b>1\right],\qquad \left[1-\frac{v^2}{b^2}\gg\frac{1}{t}\right]$$

The inequality in the first square brackets shows that (13) is not applicable when $v \to 2b^2/(1+b^2)$ as $b>1$ since $\sigma\to\infty$ results from (13) for $A_2\to 0$. In fact, $\sigma$ remains finite and continuous in this domain. (The curve $v = 2b^2/(1+b^2)$ is shown by dot-dashes in Fig. 1).

In the saddle-point method the integral over the slit is replaced by an integral over the contour $\operatorname{Im} f$ = const passing through the saddle points. Not only the horizontal but also the vertical slits fall into this contour for $v > 2b^2/(1+b^2)$. Hence, in the case $2b^2/(1+b^2) < v \leq v_+$ Formula (13) does not yield the stress for $b<1$ because the integral of $G(k_-)$ over $(p_1, p_2)$ is no longer estimated in (13). For this same reason $\sigma$ is described by (13) with $2b^2/(1+b^2) < v < v_-$ for $b>1$. The inequalities in the second and third square brackets indicate these facts.

And finally, the fourth parenthesis means that (13) is not valid for $v\to b$. Hence, $\sigma\to\infty$, i.e. $f$ changes slightly along the line of steepest descent.

The saddle point method becomes inapplicable since the large parameter $t$ is multiplied by the small parameter $1-v^2/b^2$.

The function (13) has a maximum, propagated with velocity $v_m=(1-\delta)\sqrt{2/t}$. This maximum is equal to $(\pi e)^{-1/2} t^{-1}(1-(3/8)\delta)$. The exponent of the exponential increases in absolute value with distance from the maximum deep into the half-space, and rapidly becomes approximately $t$, hence the stresses at

$$v_m \ll x < \max\left(\left(\frac{2b^2}{1+b^2}\right),\ v_+\right)\qquad \frac{2b^2}{1+b^2}<v<v_-\ \text{ to } b>1$$

may be considered zero for large $t$.

Formulas (12) and (13), as well as the majority of the subsequent results, are generally inapplicable in case (b).

The integral along the vertical slit in (9) may also be estimated by the saddle point method, by utilizing the fact that the length of this slit is of the order of $\sqrt{\delta}$ in cases (a) and (c). Then for $\sqrt{\delta} \ll 1$, in the neighborhood of the slit, $f_\pm$ may be represented approximately as

$$f_\pm = -\frac{b^2(1-v)}{1-b^2} + z\left(1 - v\frac{1+3b^2}{4b^2}\right) \pm \frac{v(1-b^2)}{4b^2}\left(z^2 + \frac{4\delta b^8}{(1-b^2)^4}\right)^{1/2} + O(\delta) + O(z^2) \qquad (z = p - \operatorname{Re} p_3) \tag{14}$$

In this case the saddle points are easily found

$$z = \frac{\pm b^3[4b^2 - v(1+3b^2)]\sqrt{\delta}}{2(1-b^2)^2\sqrt{(v-1)(2b^2 - v(1+b^2))}} \tag{15}$$

The condition for applicability of (14) is: $z \sim \sqrt{\delta}$. It is seen from (15) that this condition is not satisfied if $v \to 2b^2/(1+b^2)$ or $v \to 1$ (more exactly: $v \to v_s$). The first member of the asymptotic expansion has the form

$$\frac{1}{2\pi i}\oint_{p_3}^{p_4} G(k_-)\,dp = \frac{\sqrt{b}\exp[-b^2(1-b^2)^{-1}\left(t - x + b^{-1}\sqrt{\delta(t-x)(2b^2t - x(1+b^2))}\right)]}{[4\delta(v-1)(2b^2 - v(1+b^2))]^{1/4}(\pi t)^{1/2}} \qquad (\delta^{1/2}t \gg 1) \tag{16}$$

It is seen from (16) that far from the acoustic wave front the exponent of the exponential is $\sim t$ in absolute value for $\sqrt{\delta} \ll 1$ and the integral may be considered practically zero for $t \gg 1$. On the other hand, (16) also shows that the greatest stresses will be in the neighborhood of the acoustic wave front. Evidently, that value of $v$ where the stress decays least, i.e. where $f$ is a maximum, is of greatest interest. Such a point of the wave for large $t$ will primarily determine the state of stress. Denoting the saddle points and the value of $f$ there by $p_n(v)$ and $f_n = f(p_n(v), v)$ and taking account of (11), we obtain

$$\frac{df_n}{dv} = \frac{df}{\partial p}\frac{\partial p_n}{\partial v} - \frac{\partial f}{\partial v} = -\frac{\partial f}{\partial v} = -k(p_n) = 0$$

to find the maximum value of $f$.

Eq. $k(p_n) = 0$ has two roots: $p_1 = 0$ and $p_2 = -b^2$. The first root yields the maximum. Let us find the expansion of the corresponding branch of $k$ (which we denote by $k_m$) in the neighborhood of the point $p = 0$

$$k_m = \frac{p}{\sqrt{1+\delta}} - \frac{\delta p^2}{2(1+\delta)^{5/2}} + O(p^3) \tag{17}$$

Hence

$$p_n = -\frac{\sqrt{1+\delta} - v}{\delta}(1+\delta)^2, \qquad f_n = -\frac{[\sqrt{1+\delta} - v]^2}{2\delta}(1+\delta)^{3/2}$$

The contribution of the horizontal slit may be neglected in the neighborhood of the maximum stress and the main term of the asymptotic expansion becomes

$$\sigma = \frac{(1+\delta)^{1/4}}{\sqrt{2\pi\delta t}}\exp\left[-\frac{(\sqrt{1+\delta} - v)^2(1+\delta)^{3/2}}{2\delta}t\right] + O\left(\frac{1}{t^{3/2}}\right) \tag{18}$$

It is seen from (18) that the maximum stress decays $\sim 1/\sqrt{t}$ and is propagated with velocity $\sqrt{1+\delta}$ for all $b$. The mode of the peak stress is also independent of $b$.

To estimate the state of stress at the acoustic wave front in case (a) it is convenient to utilize (8). In case (c) the integral of $G(k_-)$ over $(p_3, p_4)$ may be replaced by the corresponding integral of $G(k_+)$. The contour of integration in the latter may be transformed into the neighborhood of the infinitely distant point, $p = 0$ of case (c) is not a branch point of $k_4$, which is hence analytic outside the slit $(p_3, p_4)$, i.e. in case

(c) we can write

$$\frac{1}{2\pi i}\oint_{p_1}^{p_4} G(k_-)\,dp = -\operatorname*{res}_{p=\infty} G(k_+) \tag{19}$$

Utilizing (8) and (19), and neglecting the contribution of the horizontal slit, an expression for $\sigma$ as a series in $\delta$ may be obtained for cases (a) and (c), in which the first two members are

$$\sigma = -\frac{b^2}{1-b^2}\exp\left[-\frac{b^2(t-x/v_s)}{1-b^2}\right]\left[\eta\left(t-\frac{x}{v_s}\right)-\eta\left(t-\frac{x(1+b^2)}{2b^2}\right)\right]\times$$

$$\times\left\{1-\delta\left[\frac{2+2b^2+b^4x}{2(1-b^2)^2}-\vartheta_s\frac{b^6(x+2)+6b^4}{2(1-b^2)^3}+\vartheta_s^2\frac{b^6}{(1-b^2)^4}\right]\right\}\qquad\left(\vartheta_s=t-\frac{x}{v_s}\right) \tag{20}$$

$$(x\delta\ll 1,\ (t-x)^2\delta\ll 1,\ t\gg 1)$$

If $x\delta \gg 1$, then all terms except those containing $\Phi x$ in the coefficients of the series in $\theta_\pm$ in (8) and (19) may be neglected. Then the series may be summed and we obtai[n]

$$\sigma = -\frac{b^2}{1-b^2}\left[\eta\left(t-\frac{x}{v_s}\right)-\eta\left(t-\frac{x(1+b^2)}{2b^2}\right)\right]\exp\left(-\frac{b^4x\delta}{2(1-b^2)^2}\right)I_0\left[\left(\frac{2x\vartheta_s\delta b^6}{(1-b^2)^3}\right)^{1/2}\right] \tag{21}$$

Expression (21) has been obtained as a result of selective summation of an alternating-sign series, and represents the solution only in a small neighborhood of the acoustic wave front.

Expressions (13), (16), (18), (20) and (21) describe the behavior of the stress for large $t$ and small $\delta$ for different $x$. The stress in the neighborhood of the acoustic wave front turns out to be most essential. By comparison, the stresses at other places may be neglected. The maximum for not very large $t$ is determined by (20), and by (18) for $t\delta \gg 1$. The mutual role of these two maxima depends on the quantity $\delta$. For very small $\delta$ (for example, $\delta \sim 10^{-6}$ for quartz), the role of the first maximum will be greatest, and for such materials the heat propagation velocity will be the essential characteristic. For not very small $\delta \sim 0.01$ to $0.1$ (metals), the second maximum will be more important, and the heat propagation velocity is an inessential characteristic. If the problem of harmonic oscillations at lower frequencies is examined (all frequencies presently accessible in the nondimensional variables utilized will refer here), then a wave is obtained which is propagated at the phase velocity $\sqrt{1+\delta}$ and is independent of $b$. Th[e] second maximum is some kind of Green's function of such a slow wave. The analysis carried out refers to cases (a) and (c); about case (b) ($1-\delta < b < 1+3\delta$ for $\delta \ll 1$) it is only known that the jumps in stress on the fronts of both waves decay very rapidly. Apparently the behavior of the stress for $t \gg 1$ is determined by (18), whose validity is indirectly confirmed by the fact that case (b) is not singular for harmonic analysis.

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